

6.2 PT

MA121...

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The Definite Integral

In section 6.1, we saw that if $f(x)$ is the antiderivative of $f'(x)$, then we have

$\int f'(x) dx = f(x) + C$ ← the indefinite integral
Integral = antiderivative.

Now we introduce the definite integral:

$$\boxed{\int_a^b f'(x) dx = [f(x)]_a^b = f(b) - f(a)}$$

Note: In a definite article, the constant, C , may be omitted.

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$$\begin{aligned} \int_0^9 (-6x + 4) dx &= \left[-6 \cdot \left(\frac{x^{1+1}}{1+1} \right) + 4x \right]_0^9 \\ &= \left[-6x \left(\frac{x^2}{2} \right) + 4x \right]_0^9 = \left[-3x^2 + 4x \right]_0^9 \\ &= [-3(9^2) + 4(9)] - [-3(0^2) + 4(0)] \\ &= -3(81) + 36 + 0 - 0 = -243 + 36 + 0 - 0 \\ &= -207. \end{aligned}$$

natural log
of 1 = 0

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$$\begin{aligned} \int_e^1 \left(\frac{1}{e^x} + \frac{1}{x} \right) dx &= \int_e^1 (e^{-x} + \frac{1}{x}) dx \\ &= \left[-\frac{e^{-x}}{1} + \ln|x| \right]_e^1 = \left[-\frac{1}{e^x} + \ln|x| \right]_e^1 \\ &= \left[-\frac{1}{e^1} + \ln 1 \right] - \left[-\frac{1}{e^e} + \ln e \right] = -\frac{1}{e} + 0 + \frac{1}{e^e} - 1 \\ &= -\frac{1}{e} + \frac{1}{e^e} - 1 \end{aligned}$$

Properties of Definite Integrals (p.184)

$$\hookrightarrow \int_a^a f(x) dx = 0$$

$$\hookrightarrow \int_a^b f(x) dx = -\int_b^a f(x) dx$$

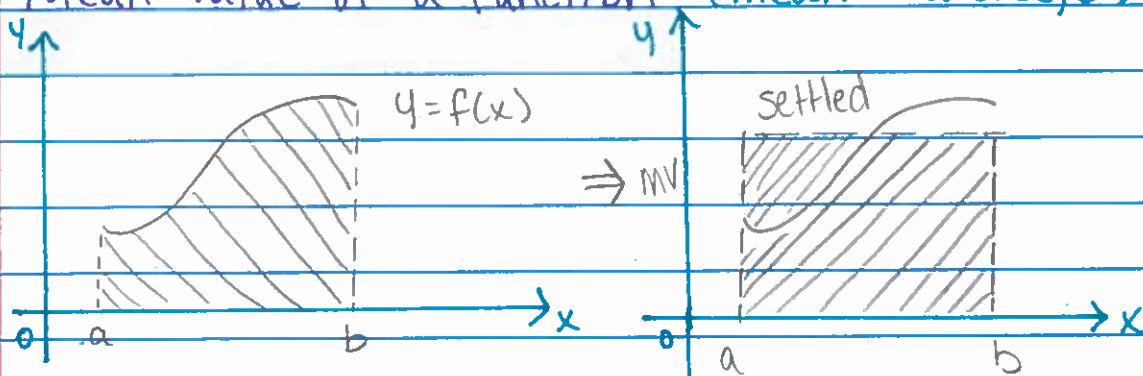
$$\hookrightarrow \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

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Mean value of a function (mean = average)

MV =
mean valueexample
of shaken
water
in
container

So we have $\int_a^b f(x) dx = (MV)(b-a)$

$$\Rightarrow MV = \frac{\int_a^b f(x) dx}{b-a}$$

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given $MC(q) = 0.00009q^2 - 0.04q + 50$, $0 \leq q \leq 500$ Currently, $q_0 = 100$ units. we want to increase by 50The new production level will be $q_1 = 150$

i) additional cost will be

$$\begin{aligned} \int_{q_0}^{q_1} MC(q) dq &= \int_{100}^{150} (0.00009q^2 - 0.04q + 50) dq \\ &= \left[0.00009 \left(\frac{q^{2+1}}{2+1} \right) - 0.04 \left(\frac{q^{1+1}}{1+1} \right) + 50q \right]_{100}^{150} \\ &= \left[0.00009 \left(\frac{q^3}{3} \right) - 0.04 \left(\frac{q^2}{2} \right) + 50q \right]_{100}^{150} \\ &= \left[0.00003q^3 - 0.02q^2 + 50q \right]_{100}^{150} \\ &= [0.00003(150)^3 - 0.02(150)^2 + 50(150)] - [0.00003(100)^3 - 0.02(100)^2 + 50(100)] \\ &= 7151.25 - 4830 = \$2321.25 \end{aligned}$$

ii) Average cost = $\frac{\int_{q_0}^{q_1} MC(q) dq}{q_1 - q_0} = \frac{\int_{100}^{150} MC(q) dq}{150 - 100}$

$$= \frac{2321.25}{50} = \$46.425$$

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Given $\int_2^5 f(x) dx = 7$ and $\int_5^9 f(x) dx = 11$

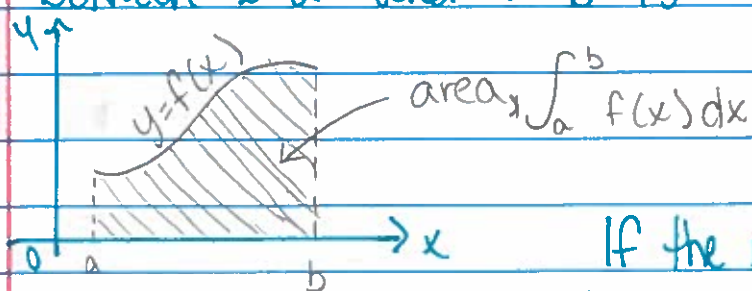
$$i) \int_5^2 f(x) dx = -\int_2^5 f(x) dx = -7$$

$$ii) \int_2^9 f(x) dx = \int_2^5 f(x) dx + \int_5^9 f(x) dx = 7 + 11 = 18$$

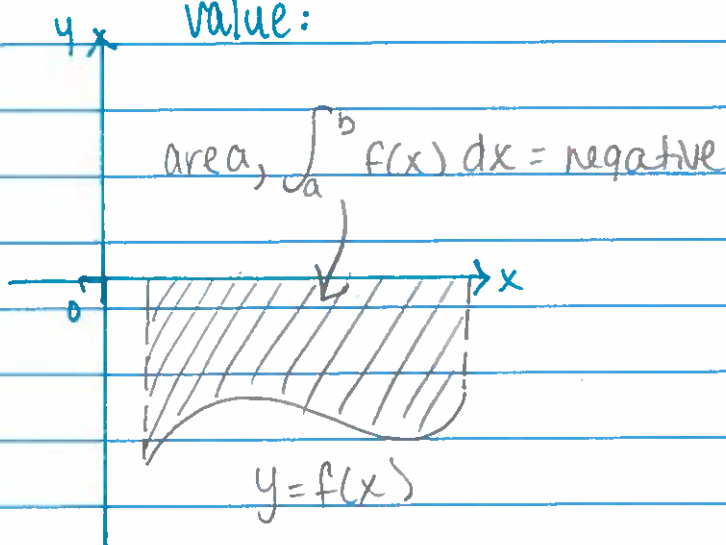
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Area

If $f(x)$ is continuous on an interval, $a \leq x \leq b$, then the area between $y=f(x)$ and the x -axis between $x=a$ and $x=b$ is



If the region is below the x -axis then its area takes a negative value:



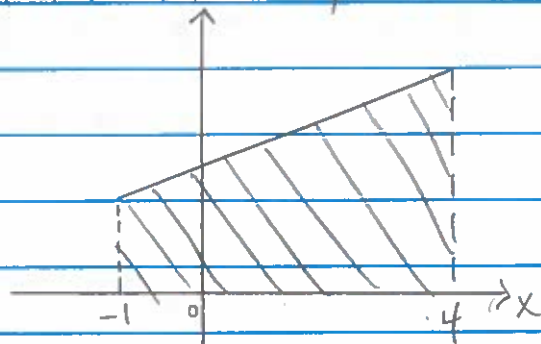
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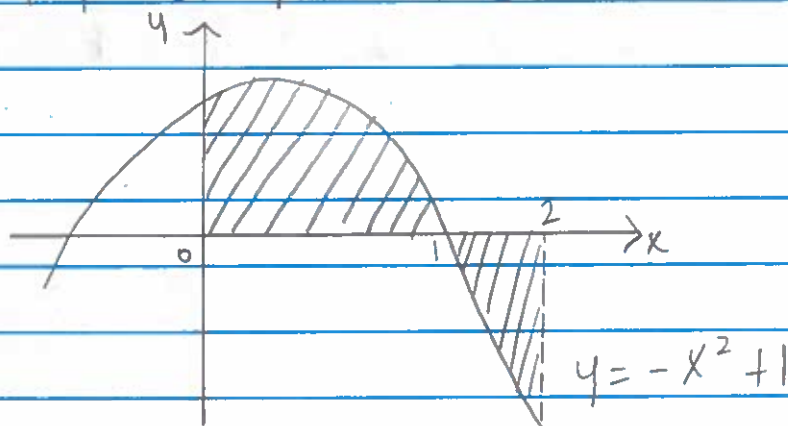
$$f(x) = 2x + 6, \quad a = -1 \text{ to } b = 4$$



$$\begin{aligned} \text{Area} &= \int_{-1}^4 f(x) dx \\ &= \int_{-1}^4 (2x + 6) dx \\ &= [x^2 + 6x]_{-1}^4 \\ &= (4^2 + 6(4)) - ((-1)^2 + 6(-1)) \\ &= (16 + 24) - (1 - 6) \\ &= 40 - (-5) \\ &= 45 \text{ unit}^2 \end{aligned}$$

Find the total area between $y = -x^2 + 1$ and the x-axis from $x = 0$ to $x = 2$.

graph of $y = x^2 + 1$ on $0 \leq x \leq 2$



Area:

$$\begin{aligned} &\int_0^1 f(x) dx + \left| \int_1^2 (x^2 + 1) dx \right| \\ &= \left[-\frac{1}{3}x^3 + x \right]_0^1 + \left| \left[-\frac{1}{3}x^3 + x \right]_1^2 \right| \\ &= \left[-\frac{1}{3}(1)^3 + 1 \right] - \left[-\frac{1}{3}(0) + 0 \right] + \left| \left[-\frac{1}{3}(2)^3 + 2 \right] - \left[-\frac{1}{3}(1)^3 + 1 \right] \right| \\ &= -\frac{1}{3} + 1 - 0 + \left| -\frac{8}{3} + 2 - \left(-\frac{2}{3} + 1 \right) \right| \\ &= \frac{2}{3} + \left| -\frac{4}{3} \right| \\ &= \frac{2}{3} + \frac{4}{3} \\ &= \frac{6}{3} = 2 \text{ units}^2 \end{aligned}$$

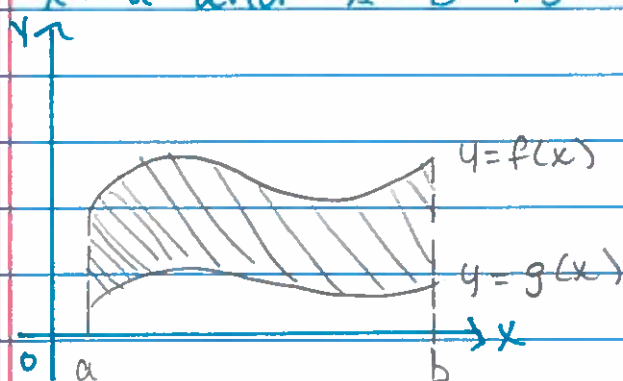
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Area bounded between two curves

If $f(x)$ and $g(x)$ are two continuous functions on $a \leq x \leq b$, and if $f(x) \geq g(x)$ on $a \leq x \leq b$, then the area between $f(x)$ and $g(x)$ between $x=a$ and $x=b$ is

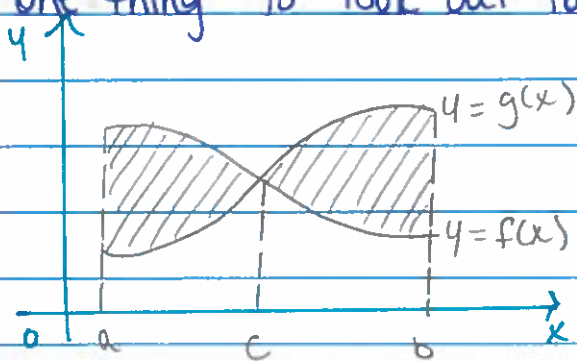


Area:

$$\int_a^b \underbrace{f(x)}_{\text{upper curve}} - \underbrace{g(x)}_{\text{lower curve}} dx$$

If you happen to take $\int_a^b (\text{lower curve} - \text{upper curve}) dx$ you will obtain a negative value for the area.

One thing to look out for:



we need to determine any intersections on the interval $a \leq x \leq b$. Let's say $x=c$ is an intersection. So,

$$\text{Area: } \int_a^c (f(x) - g(x)) dx + \int_c^b (g(x) - f(x)) dx$$

7p19) Find the area bounded by $f(x) = 2x + 5$ and $g(x) = x^2$ between $a=0$ and $b=3$.

Let's see if any intersections occur on $0 \leq x \leq 3$.

equation: $f(x) = g(x)$

$$\Rightarrow 2x + 5 = x^2$$

$$\Rightarrow x^2 - 2x - 5 = 0$$

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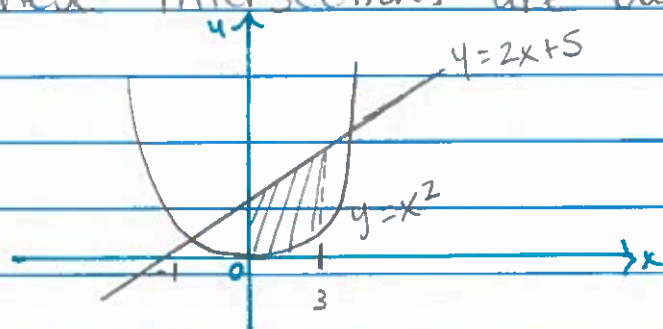
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$$\Rightarrow X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 4.89...}{2}$$

$$\Rightarrow X = \frac{-2.89}{2} \text{ or } X = \frac{6.89}{2} ; \Rightarrow X = -1.4.. \text{ or } X = 3.4...$$

These intersections are outside of $0 \leq x \leq 3$.



$$\text{Area} = \int_0^3 (2x + 5 - x^2) dx = \left[x^2 + 5x - \frac{1}{3}x^3 \right]_0^3$$

$$= \left[3^2 + 5(3) - \frac{1}{3}(3^3) \right] - 0 = 9 + 15 - 9 = 15 \text{ unit}^2$$

*good exam question

15 p 191 find the area of the region bounded by

$$f(x) = \sqrt{x} \text{ and } g(x) = x$$

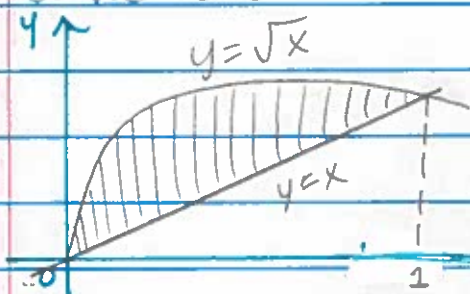
find the intersections.

$$f(x) = g(x)$$

$$\Rightarrow \sqrt{x} = x \Rightarrow x - \sqrt{x} = 0 \Rightarrow \sqrt{x}(\sqrt{x} - 1) = 0$$

$$\Rightarrow \sqrt{x} = 0 \text{ or } \sqrt{x} - 1 = 0 \Rightarrow x = 0 \text{ or } \sqrt{x} = 1 \Rightarrow x = 0 \text{ or } x = 1$$

So we have



$$\text{area} = \int_0^1 (\sqrt{x} - x) dx$$

$$= \int_0^1 (x^{1/2} - x) dx$$

$$= \left[\frac{x^{1/2+1}}{1/2+1} - \frac{x^{1+1}}{1+1} \right]_0^1$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1$$

$$= \left[\frac{2}{3} \sqrt{x^3} - \frac{1}{2}x \right]_0^1$$

$$= \frac{2}{3} \sqrt{1^3} - \frac{1}{2}(1^2) - 0$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6} \text{ unit}^2$$